Journal of Global Optimization (2005) 31: 405–420
DOI 10.1007/s10898-003-5611-6

Complementarity Problems and Variational Inequalities. A Unified Approach of Solvability by an Implicit Leray-Schauder Type Alternative

G. ISAC
Department of Mathematics, Royal Military College of Canada, P. O. Box 17000, STN Forces, Kingston, Ontario, Canada K7K 7B4

(Received 25 September 2002; accepted in revised form 29 March 2003)

Abstract. In several recent papers we obtained existence theorems for complementarity problems and variational inequalities using for each of them a particular notion of exceptional family of elements. Now, in this paper we introduce a new notion of exceptional family of elements. This notion is based on an Implicit Leray-Schauder Alternative. By this new notion we obtain a unification of the study of solvability of complementarity problems and of variational inequalities. The paper is finished with a section dedicated to variational inequalities with δ-pseudomonotone operators.

Key words: Complementarity problems and variational inequalities, Exceptional family of elements, Implicit Leray–Schauder Alternative

1. Introduction

Variational Inequalities Theory and Complementarity Theory, both have many applications in Economics, Engineering, Mechanics, Elasticity, Fluid Mechanics, Game Theory and Optimization [2, 6, 18–20, 23–25, 32, 40, 44].

Generally, the applications are related to the study of equilibrium. In this paper we will consider variational inequalities on unbounded convex sets and nonlinear complementarity problems. The study of solvability of such problems, generally defined for nonlinear mappings, is not an easy problem. Because of this fact, it is natural to apply deep and powerful topological methods.

G. Isac, V. Bulavski and V. Kalashnikov introduced in [31] a new topological method, applicable to the study of complementarity problems and also to the study of variational inequalities.

This method was introduced by using the topological degree. Until now several papers based on this method have been published [7–9, 21–23, 25–39, 47, 50–60]. Recently, we remarked that this method can be developed applying Leray–Schauder type alternatives [29–30].

We note that our topological method is based on the notion of "*exceptional family of elements*" which is supported by the topological degree or by Leray–Schauder type alternatives. This notion was initially introduced for complementarity problems and after some time it was adapted for variational inequalities [50, 39]. So, we obtained two variants of the notion of *exceptional family of elements* and by this way we obtained several existence theorems. Because a complementarity problem is a variational inequality on a closed convex cone, it is interesting to know if both notions can be unified.

This paper is dedicated to this problem. We will show that by using a special implicit Leray–Schauder type alternative we unify both notions of exceptional family of elements. By this unification, we can extend to variational inequalities several existence results obtained before for complementarity problems.

In the last part of this paper, we will show that if, a variational inequality is defined on an unbounded set by a pseudomonotone operator, then the solvability is equivalent to the fact that the mapping is without exceptional families of elements.

We will finish this paper by some comments and open problems.

2. Preliminaries

We will denote by $(H,<.,.>)$ an arbitrary Hilbert space, by $\boldsymbol{K}$ a closed convex cone in H and by Ω an arbitrary non-empty closed convex set. A closed convex cone $\boldsymbol{K} \subset H$ is a non-empty closed subset satisfying the following properties:

(1) $\boldsymbol{K} + \boldsymbol{K} \subseteq \boldsymbol{K}$,
(2) $\lambda \boldsymbol{K} \subseteq \boldsymbol{K}$, for all $\lambda \in \boldsymbol{R}_+$.

We denote by $\boldsymbol{K}^*$ the dual cone of $\boldsymbol{K}$, i.e., the set

$$K^*\{y \in H | \langle x, y\rangle \geq 0,\ for\ all\ x \in K\}.$$

It is known that $\boldsymbol{K}^*$ is a closed convex cone [23]. If $\Omega \subset H$ is an arbitrary closed convex set (or in particular $\Omega = \boldsymbol{K}$) then for any $x \in H$, the projection $P_\Omega\ (x)$ of x onto Ω is unique and it is well defined. We have that $P_\Omega(x)$ is the unique element in Ω such that

$$\|x - P_\Omega(x)\| = \min_{y \in \Omega} \|x - y\|.$$

In particular, if $\boldsymbol{K} \subset H$ is a closed convex cone and $x \in H$ is an arbitrary element, we denote by $P_\Omega(x)$ the projection of x onto $\mathbf{K}$. We recall the following classical result.

Lemma 2.1. *(i) If $\Omega \subset H$ is an arbitrary closed convex set, then for any $x \in H$, P_Ω (x) is the unique element in Ω such that:*

$$\langle P_\Omega(x) - x, P_\Omega(x) - y\rangle \leq 0,\ for\ all\ y \in \Omega$$

(ii) If $\boldsymbol{K} \subset H$ is a closed convex cone, then for every $x \in H$, P_K (x) is the unique element in $\boldsymbol{K}$ such that:

(1) $\langle P_K(x) - x, y\rangle \geq$), *for all* $y \in \boldsymbol{K}$
(2) $\langle P_K(x) - x, P_K(x)\rangle = 0$

Proof. For a proof of this result, the reader is referred to [2]. □

If $\Omega \subset H$ is a closed convex set and $x_* \in \Omega$, then the normal cone of the set Ω at the point x_* is by definition

$$N_\Omega(x_*) = \{\xi \in H | \langle \xi, y - x_*\rangle \leq 0, \ for \ all y \in \Omega\}$$

We have the following result.

Proposition 2.2 (10). *If* $\Omega \subset H$ *is a closed convex set and* $x \in H$ *is an arbitrary element, then we have that* $y = P_\Omega(x)$*if and only if* $x \in y + N_\Omega(y)$. □

We need also to recall the following notions. We say that a mapping T: $H \to H$ is completely continuous if T is continuous and for any bounded set $B \subset H$, we have that $T(B)$ is relatively compact. A completely continuous field on H is a mapping f:$H \to H$ of the form $f(x) = x - T(x)$, for any $x \in H$, where T: $H \to H$ is a completely continuous mapping. If $\boldsymbol{K} \subset H$ is a closed convex cone, we call a mapping f: $\boldsymbol{K} \to H$ regular, if for each sequence $\{x_n\}_{n\in N} \subset \boldsymbol{K}$, weakly convergent to x_* and such that the sequence converges to $v_* \in H$ in norm, then the equation $f(x_*) = \upsilon_*$ holds.

3. Complementarity problems and variational inequalities. Two notions of exceptional family of elements

Let $(H, < ., . >)$ be a Hilbert space, $\boldsymbol{K} \subset H$ a closed convex cone and f: $\boldsymbol{K} \to H$ a mapping.

The Nonlinear Complementarity Problem defined by f and the cone $\boldsymbol{K}$ is

$$NCP(f, K) : \begin{cases} \text{find } x_* \in K & \text{such that} \\ f(x_*) \in K^* & \text{and } \langle x_*, f(x_*)\rangle = 0. \end{cases}$$

If f is an affine mapping that is, $f(x) = Ax + b$, where A: $H \to H$ is a linear operator and b is an element in H, then the problem $NCP(f, \boldsymbol{K})$ is the linear complementarity problem and it is denoted by $LCP(A, b, \boldsymbol{K})$. *The problem* $LCP(A, b, \boldsymbol{K})$ *have been studied by many authors.*

We consider in this paper only the problem $NCP(f, \boldsymbol{K})$. *The following definition was considered in several of our papers: [7, 8, 25, 27, 29].*

Definition 3.1. We say that a family of elements $\{x_r\}_{r>0} \subset \boldsymbol{K}$ is an exceptional family of elements for f, with respect to $\boldsymbol{K}$, if for every real number $r > 0$, there exists a real number $\mu_r > 0$ such that the vector $u_r = \mu_r x_r + f(x_r)$ satisfies the following conditions:

(1) $u_r \in \boldsymbol{K}^*$,
(2) $\langle u_r, x_r \rangle = 0$,
(3) $\|x_r\| \to +\infty$ *as* $r \to +\infty$.

Definition 1 is justified by the following alternative result.

Theorem 3.1. *Let be a Hilbert space* $\boldsymbol{K} \subset H$ *a closed convex cone and* $f:H\to H$ *a completely continuous field.*

Then there exists either a solution to the problem NCP(f, $\boldsymbol{K}$*) or f has an exceptional family of elements with respect to* $\boldsymbol{K}$*. (in the sense of Definition 1).*

Proof. A proof of this result is in paper [29]. □

Corollary 3.1. *Let be a Hilbert space* $\boldsymbol{K} \subset H$ *a closed convex cone and* $f:H\to H$ *a completely continuous field. If f is without exceptional family of elements, in the sense of Definition 1, with respect to* $\boldsymbol{K}$*, then the problem NCP(f,* $\boldsymbol{K}$*) has a solution.*

Remark 3.2. In a recent paper [9], A. Carbone and P. P. Zabreiko obtained a variant of *Theorem 3* for the case when f is a regular mapping and satisfying the property that for any bounded set $B \subset H$, $f(B)$ is relatively compact. In this case f is not necessarily a completely continuous field.

Now, we suppose that $\Omega \subset H$ is an arbitrary unbounded closed convex set and $f:H\to H$ a completely continuous field. We suppose that f has the representation $f(x) = x - T(x)$, for any $x \in H$, and we consider the variational inequality defined by f and Ω, i.e.,

$$VI(f,\Omega : \begin{cases} \text{find } x_* \in \Omega & \text{such that} \\ \langle f(x_*), x - x_* \rangle & \geq 0 \text{for all } x \in \Omega \end{cases}$$

In our paper [39] we introduced the following definition.

Definition 3.2. We say that $\{x_r\}_{r>0} \subset H$ is a exceptional family of elements for the completely continuous field, f, with respect to Ω, if the following conditions are satisfied:

(1) $\|_r\| \to +\infty$ *as* $r \to \infty$,
(2) for any $r>0$ there exists a real number $\mu_r > 1$ such that $\mu_r x_r \in \Omega$ and $T(x_r)$-$\mu_r x_r \in N_\Omega(\mu_r x_r)$.

We have the following result.

Remark 3.3. If in *Definition 3.2* we have that Ω is a closed convex cone and $\{x_r\}_{r>0}$ is an *EFE* then $\{x_r\}_{r>0}$ is an *EFE* in the sense of *Definition 3.1* too. Because in *Definition 3.2* we have that $\mu_r x_r \in \Omega$, we can not transfer some results obtained for complementarity problems to general variational inequalities. To solve this problem we will introduce in **Section 5** another notion of *EFE* based on an *Implicit Leray–Schauder Alternative.*

Theorem 3.4. *Let (H,<.,.>) be a Hilbert space, $\Omega \subset H$ an arbitrary unbounded closed convex set and f:H→ H a completely continuous field. Then the problem VI(f, Ω)has at least one of the following properties:*

(1) VI(f, Ω)has a solution,
(2) the completely continuous field f has an exceptional family of elements in the sense of Definition 3.2, with respect to Ω.

Proof. A proof of this theorem is in [39]. □

Corollary 3.2. *Let (H,<.,.>) be a Hilbert space, $\Omega \subset H$ an arbitrary unbounded closed convex set and f:H→ H a completely continuous field. If f is without exceptional family of elements, in the sense of Definition 3.2, with respect toΩ, then the problem VI(f, Ω)has a solution.* □

In our papers [7, 8, 26–36, 38, 39] and in [47, 50–60] are shown classes of mappings without exceptional family of elements in the sense of *Definition 3.1 or Definition 3.2*. But recently we remarked that some properties of mappings related to *Definition 3.1* that is related to complementarity problems couldn't be extended to an arbitrary variational inequality. Because of this fact, a natural question is to know if it is possible to unify the both notions of exceptional family of elements.

In the next sections of this paper, we will show that this unification is possible by an implicit Leray–Schauder type alternative.

4. An implicit Leray-Schauder type alternative

It is well known that one of the most important theorems of Nonlinear Functional Analysis is the Leray–Schauder Alternative, proved in 1934 by the topological degree. [46].

Now, there exist several kinds of Leray–Schauder type alternatives proved by different methods not based on topological degree. See [3] among others. We note that, the classical Leray–Schauder Alternative has many applications to ordinary differential equations.

First, we recall the classical Leray-Schauder Alternative but given in a Hilbert space.

Theorem 4.1. (Leray-Schauder Alternative). *Let (H,<.,.>) be a Hilbert space, $D \subset H$ a convex set and U a subset open in D such that $0 \in U$. then each continuous compact mapping $f : \bar{U} \to D$ has at least one of the following properties:*

(1) f has a fixed point
(2) there is $(x_*, \lambda_*) \in \partial U \times]0, 1[$ such that $x_* = \lambda_* f(x_*)$

We denoted by ∂U the boundary of U.. We note that *Theorem 6* is the explicit form of the following more general result proved by A. J. B. Potter in 1972 [48].

Let $E(\tau)$ be a locally convex space and $B \subset E$ a closed convex set with int(B) non-empty and such that $0 \in int(B)$.

Theorem 4.2. (Potter). *Let T: $[0, 1] \times B \to E$ be a continuous mapping such that $T([0, 1] \times B)$ is relatively compact in E. We consider on $[0, 1] \times B$ the product topology. Suppose:*

(1) *$T(t, x) \neq x$, for all $x \in \partial B$ and $t \in [0, 1]$,*
(2) *$T(\{0\} \times \partial B) \subset B$.*

Then, there is an element $x_* \in B$ such that $T(1, x_*) = x_*$.

Proof. A proof without topological degree is given in [48]. □

The following Implicit Leray-Schauder type alternative is a consequence of Theorem 4.2.

Theorem 4.3. *Let T: $[0, 1] \times B \to E$ be a continuous mapping such that $T([0,1]\times B)$ is relatively compact in E. We consider on $[0, 1] \times B$ the product topology. If the following assumptions are satisfied:*

(1) *$T(\{0\} \times \partial B) \subset B$*
(2) *$T(0, x) \neq x$ for any $x \in \partial B$, then at least one of the following properties is satisfied:*
(i) *there exists $x_* \in B$ such that $T(1, x_*) = x_*$,*
(ii) *there exists $(t_*, x_*) \in]0, 1[\times \partial B$ such that $T(t_*, x_*) = x$.* □

Proof. The reader can find easily the proof of this result using Theorem 4.2. □

Remark 4.4. If in Theorem 4.3 we consider $T(t, x) = tf(x)$, where $b = D$ and f: $\to$ D is a continuous compact mapping we obtain Theorem 4.1.

5. A New notion of exceptional family of elements

Let $(H, <., .>)$ be a Hilbert space, and f: $H \to H$ a completely continuous field with a representation of the form $f(x) = x - T(x)$, for all $x \in H$.

Let $\Omega \subset H$ be an unbounded closed convex set. We consider again the variational inequality $VI(f,\Omega)$.

Definition 5.1. We denote by $\rho = ||P_\Omega(0)||$. We say that a family of elements $\{x_r\}_{r>\rho}$ is an Exceptional Family of Elements (shortly EFE) for the completely continuous field f with respect to Ω, if the following properties are satisfied:

(1) $\|x_r\| \to \infty as r \to +\infty (r > \rho)$,
(2) for any r>ρ there exists t$_r$ ∈] *0, 1* [such that $t_r T(x) - x_r \in N_\Omega, (x_r)$.

We have the following result.

Theorem 5.1. (Alternative theorem). *Let* $\Omega \subset H$ *be a non-empty unbounded closed convex set and f:H* $\to$ *H a completely continuous field such that* $f(x) = x - T(x)$ *for any* $x \in H$. *Then there exists either a solution to the problem VI (f,Ω), or the mapping f has an EFE, in the sense of Definition 5.1, with respect toΩ.*

Proof. First, we recall that the problem *VI*(*f*,Ω) has a solution if and only if the mapping $\Phi(x) = P_\Omega(x - f(x))$ has a fixed point. If the problem *VI*(*f*,Ω) has a solution, then in this case the theorem is proved.

We suppose that the problem *VI(f*,Ω) has no solution. In this case, we consider for any real number *r*, such that $r > \rho = ||P_\Omega(0)||$, the closed convex set $B_r = \{x \in H | \|x\| \leq r\}$. Obviously $\partial B_r = \{x \in H | \|x\| = r\}$. For any $r > 0$, we consider the mapping $\Phi_r : [0,1] \times B_r \to H$ defined by $\Phi_r[t,x] = P_\Omega[t(x - f(x))] = P_\Omega[t(T(x))]$. We have that Φ_r is continuous and $\Phi_r([0,1] \times B_r)$ is relatively compact in *H*. Moreover, $\Phi_r)\{0\} \times \partial B_r) \subset B_r$ and for any $x \in \partial B_r$, we have that $\Phi_r (0, x) \neq x$.

We deduce that the assumption of *Theorem 4.3* are satisfied, and because we supposed that the problem *VI*(*f*,Ω) is without solution, we have that $\Phi_r (1, x) \neq x$ for any $x \in B_r$, which implies that there exists $t_r \in]0, 1[$ and $x_r \in \partial B_r$ such that $\Phi_r(t_r, x_r) = x_r$, for any $r > \rho$.

We have that for any $r > \rho$ there exists $t_r \in]0,1[$ and $x_r \in \partial B_r$ such that $x_r = P_\Omega[t_r(T(x_r))]$. Therefore we have $x_r \in \Omega$ and $t_r T(x_r) \in x_r + N_\Omega(x_r)$. From the last relation we obtain

$$t_r T(x_r) = x_r + \xi, \text{ where } \xi \in N_\Omega(x_r).$$

Hence, we have $t_r T(x_r) - x_r \in N_\Omega(x_r)$ and we have that the family $\{x_r\}_{r>\rho}$ is an *EFE* for the completely continuous field *f*, and the proof is complete. □

Theorem 5.2. (An Existence Theorem). *Let* Ω ⊂ *H be a non-empty unbounded closed convex set and f: H* → *H a completely continuous field. If f is without EFE in the sense of Definition 5.1, with respect to* Ω, *then the problem VI (f,* Ω*) has a solution.*

Proof. The theorem is a consequence of *Theorem 5.1.*

Corollary 5.1. *Let* ($\boldsymbol{R}^n$, <., .>) *be the n-dimensional Euclidean space,* $\Omega \subset \boldsymbol{R}^n$ *an unbounded closed convex set and f:* $\boldsymbol{R}^n \to \boldsymbol{R}^n$ *a continuous mapping. If f is without EFE, in the sense of Definition 5.1, with respect to* Ω *(considering f = I - (I - f), with I the identity mapping), then the problem VI (f,*Ω*) has a solution.* □

Remark 5.3. If $\Omega = \boldsymbol{K}$, where **K** is a closed convex cone in H, then in this case the notion of *EFE* defined in *Definition 5.1* is exactly the notion of *EFE* used in Complementarity Theory.

Indeed, let $\{x_r\}_{r>\rho}$ be an *EFE* as defined in *Definition 5.1*. In this case we have, $\rho = \|P_K(0)\| = 0$, i.e., $r > 0$ and for any $r > 0$ we have $t_r T(x_r) - x_r \in N_\Omega(x_r)$, which implies

$$\langle t_r T(x_r) - x_r, y - x_r \rangle \leq 0, \quad for\ all\ y \in K.$$

From the last inequality we deduce

$$\langle T(x_r) - \frac{1}{t_r} x_r, y - x_r \rangle \leq 0, \text{ for all } y \in K,$$

or

$$\langle x_r - T(x_r) + \left(\frac{1}{t_r} - 1\right) x_r, y - x_r \rangle \geq 0, \text{ for all } y \in K.$$

If we denote by $\mu_r = \frac{1}{t_r} - 1 > 0$ we have

$$\langle \mu_r x_r + f(x_r), y - x_r \rangle \geq 0, \text{ for all } y \in K.$$

From the last inequality we obtain

(i) $\mu_r = \mu_r x_r + f x_r) \in K^*$, *for all* $r > 0$,
(ii) $\langle \mu_r, x_r \rangle = 0$ *for all* $r > 0$,

and because $x_r \in \boldsymbol{K}$, and $||x_r|| \to +\infty$ as $r \to +\infty$ we have that $\{x_r\}_{r>0}$ is an *EFE* in the sense of *Definition 3.1*.

6. Mappings without exceptional family of elements

A consequence of *Theorem 5.2* is the fact that we must put in evidence classes of mappings without exceptional families of elements in the sense of *Definition 5.1*.

To realize this goal, we will prove, in this section, some tests which can be used as sufficient tests for the non-existence of exceptional family of elements for a given mapping. Certainly, applying these tests we obtain existence theorems for the problem $VI(f,\Omega)$.

Before doing this, we will give an equivalent form of the notion of exceptional family of elements.

Let $(H, <.,.>)$ be a Hilbert space and $\Omega \subset H$ an arbitrary unbounded closed convex set. We denote by $\rho = \|P_\Omega(0))\|$.

Definition 6.1. Given a mapping $f : H \to H$, we say that a family of elements $\{x_r\}_{r>\rho} \subset \Omega$ is an exceptional family of elements (denote by EFE) for f with respect to Ω if and only if the following conditions are satisfied:

(i) $\|x_r\| + \infty$ *as* $r \to +\infty$ *and*
(ii) *for any* $r > \rho$, *there exists* $t_r \in]0, 1[$ *such that,* $-f(x_r) - \left(\frac{1}{t_r} - 1\right) x_r \in N_\Omega(x_r)$.

Proposition 6.1. *If* $f : H \to H$ *is a completely continuous field with a representation of the form* $f(x) = x - T(x)$ *for all* $x \in H$*, then a family of elements,* $\{x_r\}_{r>\rho} \subset \Omega$ *is an exceptional family of elements in the sense of Definition 5.1, if and only if it is an exceptional family of elements for f, in the sense of Definition 6.1.*

Proof. Indeed, suppose that is an *EFE* for *f* in the sense of Definition 5.1. Then in this sense we have

(i) $\|x_r\| + \infty$ *as* $r \to +\infty$ *and*
(ii) *for any* $r > \rho$, *there exists* $t_r \in]0, 1[$ *such that,* $t_r T(x_r) - x_r \in N_\Omega(x_r)$.

We have,

$$T(x_r) - \frac{1}{t_r} x_r \in \frac{1}{t_r} N_\Omega(x_r) \subseteq N_\Omega(x_r)$$

which implies,

$$-f(x_r) - \left(\frac{1}{t_r} - 1\right) x_r \in N_\Omega(x_r)$$

Therefore $\{x_r\}_{r>\rho}$ is an *EFE* for *f* in the sense of *Definition 6.1.*
Conversely, let $\{x_r\}_{r>\rho}$ be an *EFE* for *f* in the sense of *Definition 6.1.*

Then we have,

(1) $\|x_r\| \to +\infty$ as $r \to +\infty$ and
(2) for any $r > \rho$, there exists $t_r \in]0, 1[$ such that $-f(x_r) - \left(\frac{1}{t_r} - 1\right) x_r \in N_\Omega(x_r)$.

We deduce that,

$$T(x_r) - \frac{1}{t_r} x_r \in \frac{1}{t_r} N_\Omega(x_r),$$

and finally

$$t_r T(x_r) - x_r \in t_r N_\Omega(x_r) \subseteq N_\Omega(x_r), \tag{1}$$

that is, $\{x_r\}_{r>\rho}$ is an *EFE* for *f* in the sense of *Definition 5.1.* □

In our papers [26] and [33] we introduced *condition (θ)* and we used this condition in Complementarity Theory [26, 33, 28, 29, 35, 36]. Now we consider this condition for variational inequalities.

Definition 6.2. We say that $f : H \to H$ satisfies condition (θ) with respect to a closed unbounded convex set$\Omega \subset$ H if and only if there exists $\rho_* > 0$ such that for any x $\in \Omega$ with $||x|| > \rho_*$, there exists y $\in \Omega$ with $||y|| < ||x||$ such that $\langle x - y, f(x)\rangle \geq 0$.

We note that *condition* (θ) is satisfied in several interesting situations. We indicate some of these situations.

Definition 6.3. We say that $f : H \to H$ satisfies the weak Karamardian Condition on Ω, if there exists a bounded set D $\subset \Omega$ such that for all x $\in \Omega$ D there exists y $\in$ D such that $\langle x - y, f(x)\rangle \geq 0$.

Remark 6.2. The classical Karamardian Condition [25] is when *D* is compact and $\langle x - y, f(x)\rangle > 0$.

Proposition 6.3. *If satisfies the weak Karamardian Condition, then f satisfies condition (θ).*

Proof. The proposition is a consequence of Definition 6.2 and 6.3 □

Remark 6.4. The converse of *Proposition 6.3* is not true. See our paper [26]. Another condition with applications to the study of existence of solutions to variational inequalities is the *Harker-Pang Condition* [20].

Definition 6.4. We say that $f : H \to$ satisfies Condition (HP), with respect toΩ if there exists an element $x_* \in \Omega$ such that the set $\Omega(x_*) = \{x \in \Omega | \langle f(x), x - x_*\rangle < 0\}$ is bounded or empty.

Remark 6.5. In the classical *Harker-Pang Condition*, the set $\Omega(x_*)$ is supposed to be compact or empty, which is more restrictive as in *Definition 6.4*.

Proposition 6.6. If $f : H \to H$ satisfies Condition (HP), then f satisfies **condition** (θ).

Proof. Indeed, if f satisfies Condition (HP), then there exists $x_* \in \Omega$ such that the set $\Omega(x_*)$ is bounded or empty. In this case, there exists $\rho_0 > 0$ such that $\Omega(x_*) \subset B(0, \rho_0)$, where

$$B(0, \rho_0 @) = \{x \in H | \|x\| \leqslant \rho_0\}.$$

We take ρ_* max$\{\rho_0, \|x_*\|\}$. If $x \in \Omega$ is an arbitrary element such that $\|x\| > \rho_*$, then we have $x \notin \Omega(x_*)$, which implies that $\langle f(x), x - x_*\rangle \geq 0$.

Obviously, if for any $x \in \Omega$ such that $\|x\| > \rho_*$, we take $y = x_*$ we obtain that *f* satisfies condition (θ). □

Theorem 6.7. *If $f : H \to H$ satisfies Condition (θ) with respect to an unbounded closed convex set $\Omega \subset H$, then f is without exceptional family of elements, in the sense of Definition 6.1, with respect to Ω.*

Proof. Suppose that *f* has an *EFE* $\{x_r\}_{r>\rho}$, in the sense of Definition 6.1, with respect to Ω. We recall that $\{x_r\}_{r>\rho} \subset \Omega$.

Since $\|x_r\| \to +\infty$ we take x_r with $\|x_r > \max\{\rho_o, \rho_*\}$, where $\rho_* > 0$ is the real number considered in Condition (θ).

For this x_r there exists $y_r \in \Omega$ such that $\|y_r\| < \|x_r\|$ and $\langle x_r - y_r, f(x_r)\langle \geq 0$.

We have also,

$$-f(x_r) - \left(\frac{1}{t_r} - 1\right) x_r \in N_\Omega(x_r).$$

If we denote by $\mu_r = \frac{1}{t_r} - 1$, we have $\mu_r > 0$ and $-f(x_r) - \mu_r x_r = \xi \in N_\Omega(x_r)$.

We deduce

$$\begin{aligned} 0 &\leq \langle x_r - y_r, f(x_r)\rangle = \langle x_r - y_r, -\mu_r x_r - \xi\rangle \\ &= \langle x_r - y_r, -\xi\rangle - \mu_r\langle x_r - y_r, x_r\rangle = -\langle x_r - y_r, \xi\rangle - \mu_r\langle x_r - y_r, x_r\rangle \\ &= \langle y_r - x_r, \xi\rangle - \mu_r\langle x_r - y_r, x_r\rangle \leq -\mu_r[\|x_r\|^2 - \langle y_r, x_r\rangle] \\ &< 0 \end{aligned}$$

which is a contradiction and the proof is complete. □

Corollary 6.1. *If $f : H \to H$ is a completely continuous field which satisfies Condition (θ) with respect to an unbounded, closed convex set$\Omega \subset H$, then the problem VI (f,Ω) has at least one solution.*

Corollary 6.2. *Let (R^n, <., .>) be n-dimensional Euclidean space and f: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ a continuous mapping. If f satisfies Condition (θ) or in particular, Karamardian Condition or Condition (HP), with respect to an arbitrary unbounded closed convex set$\Omega \subset \boldsymbol{R}^n$, then the problem VI (f,Ω) has at least one solution.*

Remark 6.8. *Other classes of mappings studied in Complementarity Theory and satisfying Condition (θ) can be considered now in the theory of variational inequalities. For such classes of mappings the reader is referred to [25, 26, 28–30, 33–36, 38].*

7. Variational Inequalities with Pseudomonotone Operators and the Notion of Exceptional Family of Elements

A natural problem is to ask, if there exists at least a class of mappings $\mathcal{F}$ such that if $f \in \mathcal{F}$ and the problem *VI(f, Ω)* has a solution for an arbitrary unbounded closed convex $\Omega \subset H$, then *f* is without *EFE*, in the sense of *Definition 6.1*. This section is dedicated to this problem. We will show that a such class of mappings is the class of *δ-pseudo-monotone mappings* which generalizes the class of pseudomonotone mappings, in Karamardian's sense. It is well known that, the monotone operators have been considered by many authors in the theory of variational inequalities [2, 20, 23, 24, 25, 44, 45].

The class of pseudomonotone operators was introduced in relation with some applications in Economics. Any monotone operator is pseudomonotone, but the converse is not true. For more information and results about pseudomonotone operators, the reader is referred to [1, 4–6, 11, 12–15, 16–19, 41–43, 49].

We recall the definition of a pseudomonotone operator as it was introduced by S. Karamardian [41, 42].

Let $(H, <., .>)$ be a Hilbert space, $f\colon H\to H$ a mapping, and $\Omega \subset H$ an unbounded closed convex set.

We recall that f is pseudomonotone (in Karamardian's sense) on Ω, if and only if, for any $x, y \in \Omega$, we have that $\langle x - y, f(y)\rangle \geq 0$ implies $\langle x - y, f(x)\rangle \geq 0$.

Definition 7.1. We say that f is δ-pseudomonotone onΩ if for any x$\in$ Ω there exists a real number $\delta(x) > 0$ such that for any y $\in \Omega$ with $||y|| > \delta(x)$, we have that $\langle x - y, f(y)\rangle \geq 0$ implies $\langle x - y, f(x)\rangle \geq 0$.

Remark 7.1. If f is pseudomonotone then it is δ-pseudomonotone but the converse is not true. In the following results we will use the notion of *EFE* in the sense of *Definition 6.1.*

Theorem 7.2. *Let $(H, <., .>)$ be a Hilbert space $f\colon H\to H$ a mapping and $\Omega \subset H$ an unbounded closed convex set. If f is δ-pseudomonotone on Ω and the problem VI(f,Ω)has a solution, then f is without EFE with respect to Ω.*

Proof. Indeed, let $x_* \in \Omega$ be a solution to the problem *VI(f,Ω)*. Then we have

$$\langle x - x_*, f(x_*)\rangle \geq 0 \text{ for all } x \in \Omega$$

In particular we have $\langle x - x_*, f(x_*)\rangle \geq 0$, for all $x\in \Omega$ with $\|x\| > \max(\|x_*\|, \delta(x_*), \rho)$, where $\rho = \|P_\Omega(0)\|$.

We suppose that $\{x_r\}_{r>\rho}$ is an *EFE* for f with respect to Ω. We take x_r, with $r > \rho$ and such that $\|x_r\| > \max(\|x_*\|, \delta(x_*), \rho)$. We have $-f(x_r) - \mu_r x_r = \xi \in N_\Omega(x_r)$ and we obtain (considering the δ-pseudomonotonicity)

$$\begin{aligned} 0 \ &\leq\ \langle x_r - x_*, f(x_r)\rangle = \langle x_r - x_*, -\mu_r x_r - \xi\rangle = \langle x_r - x_*, -\xi\rangle - \mu_r \langle x_r - x_*, x_r\rangle \\ &=\ \langle x_* - x_r, \xi\rangle - \mu_r\langle x_r - x_*, x_r\rangle \leqslant -\mu_r\langle x_r - x_*, x_r\rangle = -\mu_r\left[\|x_r\|^2 - \langle x_*, x_r\rangle\right] \\ &<\ 0 \end{aligned}$$

(since $\mu_r = \frac{1}{t_r} - 1 > 0$), which is a contradiction. Therefore f is without *EFE* with respect to Ω and the proof is complete. □

From **Theorem 7.2** *we deduce the following consequence.*

Corollary 7.1. *Let* $(H, <.,.>)$ *be a Hilbert space, f: H→ H a completely continuous field and* $\Omega \subset H$*, an unbounded closed convex set. If f is* δ*-pseudomonotone (in particular pseudomonotone) on*Ω*, then the problem VI(f,*Ω*)has a solution, if and only if f is without EFE, with respect to*Ω*.*

8. Comments and open problems

We presented in this paper a notion of *EFE*, which unifies the study of solvability of Complementarity Problems and of Variational Inequalities on unbounded closed convex sets in arbitrary Hilbert spaces. This new notion of *EFE* is based on an **Implicit Leray-Schauder type Alternative.**

The following questions are consequences of the results presented in this paper.

(I) It is interesting to find other classes of mappings without *EFE* in the sense of *Definition 4*.

(I) It is interesting to find other classes of mappings *f* with the property that the solvability of the problem *VI*(*f*,Ω) implies that *f* is without *EFE* with respect to Ω.

(I) Perhaps, the existence of no *EFE* property for *f* is the most general up to date coercivity property.

Acknowledgement

The author is grateful to anonymous referees for their valuable comments and improvements.

References

1. Aussel, D. (1998) Subdifferential properties of quasiconvex and pseudoconvex functions: A unified approach, *J. Opt. Theory Appl.* 97, 29–45.
2. Baiocchi, C. and Capelo, A. (1984) Variational and Quasivariational Inequalities, Applica-tions to Free-Boundary Problems,*John Wiley and Sons*.
3. Ben-El-Mechaiekh, H. and Idzik, A. (1994) A Leray-Schauder type theorem for approximable maps, *Proc. Amer. Math. Soc.* 122 Nr. 1, 105–109.
4. Bianchi, M. and Schaible, S. (1996) Generalized monotone bifunctions and equilibrium problems, *J. Opt. Theory Appl.* 90, 31–43.
5. Bianchi, M., Hadjisavvas, N. and Schaible, S. (1997) Vector equilibrium problems with generalized monotone bifunctios,*J. Opt. Theory Appl.* 92, 531–546.
6. Blum, E, and Oettli, W. (1993) Form optimization and variational inequalities to equilibrium problems, *The Mathematics Student*, 63, 1–23.
7. Bulavski, V. A., Isac, G. and Kalashnikov, V. V. (1998) Application of topological degree to complementarity problems, *In: Multilevel Optimization: Algorithms and Applications* (Eds. A. Migdalas, P. M. Pardalos and P. Warbrant) Kluwer Academic Publishers, 333–358.

8. Bulavski, V. A., Isac, G. and Kalashnikov, V. V. (2001) Application of topological degree theory to semi-definite complementarity problem,*Optimization*, 49, (2001), 405–423.
9. Carbone, A. and Zabreiko, P. P. (2002) Some remarks on complementarity problems in a Hilbert space, *(Preprint)*.
10. Clarke, F. H. (1983) Optimization and Nonsmooth Analysis, *Wiley-Interscience*, Wiley.
11. Cottle, R. W. and Yao, J. C. (1992) Pseudo-monotone complementarity problems in Hilbert spaces, *J. Opt. Theory Appl.* 75 Nr. 2 , 281–295.
12. Crouzeix, J. P. (1997) Pseudomonotone variational inequality problems: existence of solutions, *Math. Programming*, 78, 305–314.
13. Crouzeix, J. P. and Ferland, J. A. (1996) Criteria for differentiable generalized monotone maps, *Math. Programming*, 75, 399–406.
14. Daniilidis, A. and Hadjisavvas, N. (1999) On the subdifferentials of quasiconvex and pseudo-convex functions and cyclic monotonicity, *J. Math. Anal. Appl.* 237, 30–40.
15. Daniilidis, A. and Hadjisavvas, N. (1999) Characterization of nonsmooth semistrictly quasi-convex and strictly quasiconvex functions, *J. Opt. Theory Appl* 102 Nr. 3, 525–536.
16. Gowda, M. S. (1990) Affine pseudomonotone mappings and the linear complementarity problems, *SIAM J. on Matrix Anal. Appl.* 11, 373–380.
17. Hadjisavvas, N. and Schaible, S. (1996) Quasimonotone variational inequalities in Banach spaces, *J. Opt. Theory Appl.*, 90 Nr. 1, 95–111.
18. Hadjisavvas, N. and Schaible, S. (1998) From scalar to vector equilibrium problems in the quasimonotne case, *J. Opt. Theory Appl.*, 96 Nr. 2, 297–309.
19. Hadjisavvas, N. and Schaible, S. Quasimonotonicity and pseudomonotonicity in variational inequalities and equilibrium problems, *In: Generalized Convexity and Monotonoicity*: Recent Developments, (Eds. J. P. Crouzeix, J. E. Martinez-Legaz and M. Volle), Kluwer Academic Publishers (to appear).
20. Harker, P. T. and Pang, J. S. (1990) Finite– dimensional variational inequality and non-linear complementarity problems: a survey of theory, algorithms and applica-tions, *Math. Programming*, 48, 161–220.
21. Huang, Z. (2001) Generalization of an existence theorem for variational inequality problems, *Preprint, Academia SINICA, Beijing, China.*
22. Huang, N. J., Gao, C. J. and Huang, X. P. (2001) Exceptional family of elements and feasibility for nonlinear complementarity problems,*Preprint, Academia SINICA, Beijing, China.*
23. Hyers, D. H., Isac, G. and Rassias, Th. M. (1997) Topics in Nonlinear Analysis and Applications, *World Scientific*, Singapore, New Jersey, London, Hong Kong.
24. Isac, G. (1992) Complementarity Problems, *Lecture Notes in Math.*, Springer-Verlag, Nr 1528.
25. Isac, G. (2000) *Topological Methods in Complementarity Theory*, Kluwer academic Publishers.
26. Isac, G. (1999) a generalization of Karamardian's condition in complementarity theory, *Nonlinear Analysis Forum*, 4, 49–63.
27. Isac, G. (2000) Exceptional family of elements, feasibility, solvability and continuous path of ϵ-solutions for nonlinear complementarity problems,*In: Approximation and Complexity in Numerical Optimization*: Continuous and Discrete Problems, (Ed, Panos M. Pardalos), Kluwer Academic Publishers, 323–337.
28. Isac, G. On the solvability of multivalued complementarity problem: A Topological method, *Fourth European Workshop on Fuzzy Decision Analysis and Recognition Technology*, EF-DAN"99, Dorthmund, Germany, June 14–15, (Ed. R. Felix), 51–66.
29. Isac, G. (2001) Leray-Schauder type alternatives and the solvability of complementarity problems, *Topological Meth. Nonlinear Analysis 18*, 191–204.
30. Isac, G. (2000) On the solvability of complementarity problems by Leray-Schauder type alternatives, *Libertas Mathematica*, 20, 15–22.
31. Isac, G., Bulavski, V. A. and Kalashnikov, V. V. (1997) Exceptional families, topological degree and complementarity problems, *J. Global Opt.* 10, 207–225.

32. Isac, G., Bulavski, V. A. and Kalashnikov, V. V. *Complementarity, Equilibrium, Efficiency and Economics*, Kluwer Academic Publishers (Forthcoming).
33. Isac, G. and Carbone, A. (1999) Exceptional families of elements for continuous functions: Some applications to complementarity theory, *J. Global Opt.* 15, 181–196.
34. Isac, G. and Cojocaru, M. G. Functions without exceptional family of elements and the solvability of variational inequalities on unbounded sets, *(Forthcoming, Topological Meth. Nonlinear Analysis).*
35. Isac, G. and Kalashnikov, V. V. (2001) Exceptional families of elements, Leray-Schauder alternative, pseudomonotone operators and complementarity, *J. Opt. Theory Appl.*, 109 Nr. 1, 69–83.
36. Isac, G. and Kalashnikov, V. V. Exceptional families of elements, Leray-Schauder alternative and the solvability of implicit complementarity problems, *Annals of Oper. Res*. (Forthcoming).
37. Isac, G., Kostreva, M. M. and Polyashuk, M. (2001) Relational complementarity problem, *From local to Global Optimization*, A. Migdalas et al (eds.), Kluwer Academic Publishers, 327–339.
38. Isac, G. and Obuchowska, W. T. (1998) Functions without exceptional families of elements and complementarity problems, *J. Opt. Theory Appl.*, 99, 147–163.
39. Isac, G. and Zhao, Y. B. (2000) Exceptional families of elements and the solvability of variational inequalities for unbounded sets in infinite dimensional Hilbert spaces, *J. Math. Anal. Appl.* 246, 544–556.
40. Kalashnikov, V. V. (1995) Complementarity Problem and the Generalized Oligopoly Model, Habilitation Thesis, CEMI, Moscow.
41. Karamardian, S. (1976) Complementarity problems over cones with monotone and pseudomonotone maps, *J. Opt. Theory Appl*, 18, 445–454.
42. Karamardian, S. and Schaible, S (1990) Seven kinds of monotone maps, *J. Opt. Theory Appl.*, 66, 37–46.
43. Karamardian, S., Schaible, S. and Crouzeix, J. P. (1993) Characterizations of genera-lized monotone maps, *J. Opt. Theory Appl.*, 76, 399–413.
44. Kinderlehrer, D. and Stampacchia, G. (1980) An Introduction to Variational Inequalities and their Applications,*Academic Press, New York, London.*
45. Konnov, I. (2001) Combined Relaxation Methods for variational Inequalities, *Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Nr. 495.*
46. Leray, J. and Schauder. J. (1934) *Topologie et equations fonctionnelles*, Ann. Sci. Ecole Norm. Sup. 51, 45–78.
47. Obuchowska, W. T. (2001) *Exceptional families and existence results for nonlinear complementarity problem*, J. Global Opt. 19, 183–198.
48. Potter, A. J. B. (1972) An elementary version of the Leray-Schauder theorem,*J. London Math. Soc.* (2), 5 , 414–416.
49. Yao. J. Ch. (1994) Multi-valued variational inequalities with k-pseudomonotone operator, *J. Opt. Theory Appl.* , 83 Nr. 2, 391–403.
50. Zhao, Y. B. (1998) Existence Theory and Algorithms for Finite-Dimensional Variational Inequality and Complementarity Problems, *Ph.D. Dissertation Institute of Applied Mathematics, Academia SINICA, Beijing, (China).*
51. Zhao, Y. B. (1999) D-orientation sequences for continuous and nonlinear complementarity problems,*Applied Math. Computation* 106, 221–235.
52. Zhao, Y. B. and Han, J. Y. (1999) Exceptional family of elements for variational inequality problem and its applications, *J. Global Opt.* 14, 313–330.
53. Zhao, Y. B., Han, J. Y. and Qi, H. D. (1999) Exceptional families and existence theorems for variational inequality problems, *J. Opt. Theory Appl.* 101 Nr. 2, 475–495.
54. 10. Zhao, Y. B. and Isac, G. (2000) Quasi-P_* and $P(\tau, \alpha, \beta)$-maps, exceptional family of elements and complementarity problems,*J. Opt. Theory Appl.*, 105 Nr. 1 , 213–231.

55. Zhao, Y. B. and Isac, G. (2000) Properties of multivalued mapping associated with some nonmonotone complementarity problems, *SIAM J. Control Optim.,* 39 Nr. 2 , 571–593.
56. Zhao, Y. B. and Li, D. (2000) Strict feasibility conditions in nonlinear complementarity problems. *J. Opt. Theory Appl.*, 107 Nr. 2, 641–664.
57. Zhao, Y. B. and Li, D. (2001) Characterizations of a homotopy solution mapping for quasi-and semi-monotone complementarity problems, *Preprint, Academia SINCA, Beijing, (China).*
58. Zhao, Y. B. and Li, D. (2001) On a new homotopy continuation trajectory for nonlinear complementarity problems, *Math. Oper. Res.* 26 Nr. 1, 119–146.
59. Zhao, Y. B. and Sun, D. (2001) Alternative theorems for nonlinear projection equations and applications, *Nonlinear Anal.*, 46, 853–868.
60. Zhao, Y. B. and Yuan, J. Y. (2000) An alternative theorem for generalized variational inequalities and solvability of non-linear quasi-${P^M}_*$-complementarity problems, *Appl. Math. Computation,* 109, 167–182.